*Klein's Ikosaeder.**

By F. N. Cole, Ph. D.

Prof. Klein's work on the Theory of the Ikosaedron is thoroughly representative of the characteristic tendencies of the German mathematical school of which Clebsch was the founder, and in which, since 1870, Klein has held a foremost place. The wonderful ability and success of Clebsch in giving to an algebraic problem a geometrical form, in replacing complicated relations of pure quantity by properties of arrangement of a geometrical configuration, constitute the chief charm of his works for the majority of his readers. The elegance of this geometrical-algebraical method appeals to and gratifies the artistic sense, while at the same time it gives the subject a broader sweep and a deeper reality. The practical demonstration of the possibility, already theoretically evident, of interweaving and unifying two fundamentally distinct branches of mathematics was particularly appropriate and welcome at a time when the rapid development of the modern theories had led to an extreme specialization of work and study. The evident necessity of a thorough and complete investigation of the relations and applications of the various mathematical theories to each other immediately opened up a new field of mathematical research, which has been abundantly productive, and bids fair not to be exhausted for many years to come.

Clebsch's work† in this direction concerned particularly the mutual relations between the theories of the Abelian integrals and of invariants and covariants and those of the corresponding geometrical configurations. Other investigators have examined the deeper algebraical-geometrical nature of the algebraic functions.‡ The geometrical side of the theory of invariants is perhaps more

---

* Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, von Felix Klein. Leipzig, Teubner, 1884.

† An account of Clebsch's mathematical services, by his friends and pupils, appeared in the Math. Ann. Bd. VII.

‡ There belong here, first of all, the well-known theories of the representation of the algebraic functions by Riemann's surfaces, part of which, to be sure, are contemporary with Clebsch's work ; I refer, however, more especially to the investigations of which the article by Brill and Nöther, Math. Ann. Bd. VII, is a type.

conspicuous than the analytical, but it is very desirable that a purely geometrical theory of invariants, particularly of the binary forms, should be established in the way which Klein has indicated.* Of chief interest, however, for the present discussion, are the remarkable systems of relations which exist between the theory of *groups of operations*, of which the theory of substitutions constitutes a part, and nearly all other mathematical branches. It seems, indeed, as if the "Gruppentheorie" supplied, to a large extent, the essential formal structure to the various other theories, which then differ from each other only in the phase in which they are viewed. A characteristic feature of the modern mathematics is the predominant importance of theories, like that of groups of operations, which deal with discontinuous quantities. The theories which deal mainly with continuity have retreated decidedly into the background. It is a remarkable and suggestive fact that, scarcely two hundred years after the discovery of the Calculus, the higher mathematics has already exhibited a strong tendency to converge toward the oldest of all mathematical sciences, that of harmonious discontinuity—the theory of numbers.

The fundamental idea of Klein's entire mathematical work has been the investigation of a portion of this theory of operations which has a particular geometrical interest, while at the same time it is compactly united, through its analytical nature, with the entire Modern Algebra, using this name in its broadest sense. In the preface to the Ikosaedron, and in his "Vergleichende Betrachtungen über neuere geometrische Forschungen," will be found a statement by Klein himself of the programme which he had already developed as early as 1870, and which he has since accurately followed. It is the investigation of the groups of transformations of space into itself, and of those properties of space which remain unaltered—invariant—by these transformations, which has been the directing principle of his methods. Among these various transformations, those which are linear in the system of variables employed—the collineations—play a most important part. Analytically these are represented by

For space of 1 dimension
$$x_1' = \alpha x_1 + \beta x_2, \quad x_2' = \gamma x_1 + \delta x_2;$$

For space of 2 dimensions
$$\begin{aligned} x_1' &= \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3, \\ x_2' &= \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3, \\ x_3' &= \alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3; \end{aligned}$$

* Eintritts-Programm at Erlangen: Vergleichende Betrachtungen über neuere geometrische Forschungen, 1872, Note VII at the end of the book. See also Lindemann's article in Math. Ann. Bd. VII: "Ueber die Darstellung binärer Formen," etc.

For space of 3 dimensions
$$x_1' = \alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 + \delta_1 x_4,$$
$$x_2' = \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3 + \delta_2 x_4,$$
$$x_3' = \alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3 + \delta_3 x_4,$$
$$x_4' = \alpha_4 x_1 + \beta_4 x_2 + \gamma_4 x_3 + \delta_4 x_4.$$

Evidently we are not limited to any number of dimensions, but may construct in general n linear equations between any two sets of n variables each. The connection of the theory of groups of transformations with that of the invariants of Cayley is now evident. The latter are unchanged by the infinite and continuous groups of linear substitutions just mentioned, namely, by *all* the collineations of the corresponding space. From this point of view, the theory of the Cayley invariants* appears as a grand division of this greater theory, which then includes in itself the entire geometry, not only the projective, but also (and this is, as Klein has indicated, an interesting development†) the metrical.

On the other hand, Klein has studied the *discontinuous linear groups*, both those which contain a finite, and those which contain an infinite number of operations. For those who are not acquainted with this theory, I may cite, as examples of these two species of groups, the rotations of any regular polyedron which bring it to a position congruent with its initial one, and the binary group of linear substitutions $\omega' = \alpha\omega_1 + \beta\omega_2$ which gives all the periods of an elliptic function when two independent ones, ω_1 and ω_2, are known. The latter contains an infinite number of operations, which, however, are discrete (α and β being integers). Included in this latter group is also the infinite group $\omega_1' = \alpha\omega_1 + \beta\omega_2$, $\omega_2' = \gamma\omega_1 + \delta\omega_2$ when $\alpha\delta - \beta\gamma = 1$.‡ In this review of the specific portion of Klein's work contained in his "Ikosaeder," it would evidently be out of place to give any detailed account of his work as a whole, but I have thought it desirable to give the preceding fragmentary sketch, in order that the "Ikosaeder" might appear in its proper position with respect to other portions of the more general theory. Having now developed so much of the latter as is necessary for my purpose, I proceed at once to the consideration of the ikosaedron itself.

The theory of the algebraic equations, from its central position in the

---

* Cayley invariants. I have applied this name to the ordinary invariants in order to distinguish them from functions unchanged by other groups of transformations, to which the name invariant is equally applicable.

† See the Vergleichende Betrachtungen above, p. 12.

‡ See Briot and Bouquet's Theorie des Fonctions Elliptiques, p. 234.

modern mathematics, is peculiarly well adapted to serve as a base from which connection may be made with the various allied disciplines. The relation between it and the theory of substitutions is a perfect dualism, the propositions of the one being exact reproductions of those of the other; with the theory of invariants it stands by its very nature in the closest relation; and if we regard the coefficients in the equation as functions of one or more variables, the equation represents at once a geometrical configuration in space of one or more dimensions, while the discussion of the nature of the relation between the roots of the equation and its variable coefficients is the precise field of the theory of functions. It was therefore an excellent reason which led Klein, in writing a work in which the various branches of modern mathematics should be brought into a closer intimacy and mutual dependence, to select the theory of equations as a central feature. Not that this theory by any means assumes the greatest prominence in this work, but that in passing from one portion to another it always serves as a convenient stepping-stone, or a sort of central station. That exactly the equation of the fifth degree is chosen is not merely because we come upon it naturally after the solution of that of the fourth, but much more, from Klein's point of view, because the theories of the equations of the first five degrees constitute a closed system by themselves, the nature of which is most intimately connected, or rather identified, with the theory of the finite groups of linear transformations of a *single* variable. These groups present themselves, implicitly or explicitly, in every phase in which the theory of these equations can be studied, and are of such fundamental importance that Klein has preferred to devote the first portion of the work before us to a thorough exposition of their theory from all points of view, while the treatment of the general equations of the first five degrees appears in the second part as an extended application and development of this theory.

Among the various forms in which this theory of the finite linear groups of a single variable appears, the most tangible is undoubtedly that of the theory of those rotations of the ikosaedron and the kindred regular polyedra which bring these configurations to a final position which is congruent to the initial one. To this theory Klein has assigned the greatest prominence in the work before us, and he has made it the point of departure for the entire treatment of the subject. The first chapter of the book is occupied with a discussion of the groups of rotations of these regular bodies, while in the second chapter the immediate relation to the linear transformations of a single variable is exhibited by the introduction

of the stereographic projection of Riemann. Thus: the ikosaedron is supposed to be inscribed in a sphere. By projection of the ikosaedron edges from the centre of the sphere upon the spherical surface, we obtain on the latter a geometrical configuration which, for our purposes, completely replaces the ikosaedron itself. It is in the sense of this configuration, and not of the regular geometrical solid, that the name "Ikosaeder," as used by Klein, is to be understood. Obviously this results in no inconsistency with what here precedes, for all rotations which leave the ikosaedron congruent with its original position will do the same for the surface configuration. If now the sphere be placed on a horizontal plane, so that the point of tangency, which we may call the south pole of the sphere, is a vertex of the "Ikosaeder," and if then the entire configuration on the sphere be projected from the uppermost vertex, or north pole, on the plane, the resulting plane configuration furnishes valuable assistance in the further development of the theory.

To fix the position of a point on the sphere, we introduce a rectangular co-ordinate system with its origin at the centre of the sphere, and denote the co-ordinates of any point referred to this system by ξ, η, ζ. In the plane* we take another co-ordinate system, referred to two rectangular axes parallel respectively to the axes of ξ and η. The co-ordinates of any point in the plane being x and y, we denote the point by $z = x + yi$, this being the ordinary notation of the geometrical representation of complex numbers. Between the co-ordinates of any point on the sphere and those of its projection in the plane there exist the relations $x = \frac{\xi}{1-\zeta}$, $y = \frac{\eta}{1-\zeta}$, $x + iy = \frac{\xi + i\eta}{1-\zeta}$. If now the sphere undergo any rotation, including as a particular case those rotations which leave the ikosaedron congruent to its initial position, any point ξ, η, ζ will take a new position ξ', η', ζ' and at the same time its projected point z in the plane will become a new point z'. Now, we may suppose the whole of space to be rotated with the sphere, and such a rotation of space is a collineation, *i. e.* a transformation in which all points on a straight line become points on a straight line. Such transformations are denoted analytically by the system of linear equations of page 46. The effect of any rotation of the sphere is therefore to convert the point ξ, η, ζ into the point

$$\xi' = \frac{a_1\xi + b_1\eta + c_1\zeta}{d_1\xi + e_1\eta + f_1\zeta}, \quad \eta' = \frac{a_2\xi + b_2\eta + c_2\zeta}{d_1\xi + e_1\eta + f_1\zeta}, \quad \zeta' = \frac{a_3\xi + b_3\eta + c_3\zeta}{d_1\xi + d_2\eta + d_3\zeta}.$$

* The reader will notice that Klein supposes this second co-ordinate system to be projected on the sphere, thus dispensing with the plane. The "Ikosaeder" contains at the end a figure of the projection of the ikosaedron as the plane.

And now it appears that the corresponding transformation in the plane converts the point z into a new point z' which is related to the first point by the *linear* equation $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$, where α, β, γ, δ are constant quantities independent of z. This is a particular case of a more general proposition, the proof of which is not difficult. Thus there are in all ∞^{15} distinct collineations of space, namely, those which are defined by four equations of the form $x_i' = \alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 + \delta_i x_4$. These contain $16 - 1$ essential constants. Of these, those which leave a given quadric surface (in the present case a sphere) unchanged, being limited by 9 conditions, the preservation of the nine coefficients in the equation of the surface, form ∞^6. The projection of these ∞^6 transformations of the spherical surface into itself gives ∞^6 transformations in the complex plane which are defined by $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$, or by this equation together with the replacing of z by its conjugate value, which amounts to a reflection of the plane on its real axis.

Among these ∞^6 transformations of the spherical surface, we are more particularly concerned with those 120 rotations and reflections on the diametral planes which transform the ikosaedron into itself. The effect of such a rotation or reflection is to permute in a certain order all corresponding points of the ikosaedron, their projections in the plane of course undergoing the same permutations. Analytically this geometrical permutation of points is represented (in case of the rotations) by the replacement of the co-ordinate z of every point in the plane by a linear function of itself $\frac{\alpha z + \beta}{\gamma z + \delta}$. We have thus, parallel with our 60 interchanges of points, a group of 60 linear functions, these being of such a nature that if we select any one of them, and put for z successively the co-ordinates of any set of corresponding points, we get the same co-ordinates again, but in the permuted order. The completion of the theory requires the determination of these analytical expressions, which, however, need not be calculated separately, but are all obtainable by repetition and combination of two, the S and T of Klein, S denoting a rotation about the vertical axis through an angle $\frac{2\pi}{5}$, and T a rotation of 180° about the axis which bisects one of those ikosaedron edges which meet in the north pole of the sphere. For these we have* $S : z' = \varepsilon z$ and $T : z' = \frac{(\varepsilon - \varepsilon^4) z - (\varepsilon^2 - \varepsilon^3)}{(\varepsilon^3 - \varepsilon^2) z + (\varepsilon^4 - \varepsilon)} \left(\varepsilon = e^{\frac{2\pi i}{5}} \right)$. These 60 rotations combined with $z' = x' + y'i = \bar{z} = x - yi$ give the 60 reflections in addition.

---

* See Klein, p. 41.

The 60 ikosaedron rotations convert any point on the spherical surface into 60 similarly situated points, which then constitute a configuration which is unaltered, invariant, for this group of rotations. There are, however, certain of these invariant configurations which contain a less number of points. Thus the 12 vertices of the ikosaedron, the 20 centre-points of its faces and the 30 middle points of its sides are each configurations of the invariant character. To each invariant configuration belongs an invariant analytic function, namely, the analytical expression of the configuration, which is obtainable by multiplying together the values of the complex variable corresponding to the projections of the separate points of the configuration. Now, it appears that in the present case every such invariant function can be composed rationally from those three which represent the three special cases mentioned above of the ikosaedron vertices, etc., the T, H, f, of the book,* and from this it follows easily that for different values of Z, $Z : Z - 1 : 1 = H^3 : -T^2 : 1728f^5$ defines any one of the groups of 60 points which are related to each other in the way we have considered. This equation, which is of degree 60 in $\frac{z_1}{z_2}$, contains in itself implicitly the general equation of the fifth degree, for it is a resolvent of the latter. If for a given value of Z we can determine the corresponding group of 60 correlated points on the sphere, we can at once solve the equation of the fifth degree, and *vice versa.*

In the "extended group" of 120 transformations, composed of the 60 rotations above and the 60 possible reflections on planes of symmetry, the former constitute a sub-group by themselves, while the latter do not, since two reflections give a rotation, not a reflection. Moreover, if we take successively a reflection, a rotation, and the same reflection reversed, we get a rotation. *The group of the rotations is self-conjugate*† within the extended group.

In these first two chapters of the book will be found a very complete and elegant account of the theories, not only of the ikosaedron, which is identified with the theory of the equation of the fifth degree, but also of the oktaedron, the tetraedron and the "diedron," whose theories are of similar importance for the equations of the fourth and third degrees and the cyclical equations respectively. The geometrical interpretation which this theory gives to a portion of

---

* See pages 55–61.

† I use "self-conjugate sub-group" in translating Klein's "ausgezeichnete Untergruppe" and Jordan's "groupe permutable."

the theory of substitutions is well worthy of being studied, especially the geometric conception of the position of a self-conjugate sub-group within the entire group, and the resulting geometrical demonstration that the oktaedron group contains such a sub-group, while the ikosaedron group does not.

Before leaving this part of the book, one more matter deserves to be noted. If we make our 60 linear substitutions in the plane, corresponding to the 60 rotations of the sphere, homogeneous by introducing for z, $\frac{z_1}{z_2}$, writing $\begin{matrix} Z_1' = \alpha z_1 + \beta z_2, \\ Z_2' = \gamma z_1 + \delta z_2 \end{matrix}$, it appears that the "identical rotation" $z_1' = -z_1$, $z_2' = -z_2$, will appear among the homogeneous substitutions, so that there will be in all $120 = 2.60$ of these. The relation between the homogeneous and the non-homogeneous group is therefore a hemiedric isomorphism. This is not to be avoided. There is no group of binary linear substitutions which is holoedrically isomorphic with the non-homogeneous group considered. This is a fact of considerable importance for the theory of the equation of the fifth degree, as is seen later on. On this depends the appearance of an "accessory" square root in the solution.

This theory of the correspondence between groups of linear transformations of a single variable and groups of rotations of the regular polyedra receives a remarkable completion in the fifth chapter of the book. We have seen that to every group of rotations corresponds a configuration of points, to which configuration an invariant then belongs. If any point of the configuration be given, all others proceed from it by the rotations, or, if we consider the projection on the complex plane, by linear transformations, so that the entire configuration, and consequently its invariant, are at once known. The inverse of this problem would be, given the invariant Z, to find the various points z of the configuration. Only one solution of this inverse problem is necessary; *i. e.* geometrically we need find only one point of the configuration, since all others are then obtainable by known linear transformations from this one. Leaving the actual solution of this problem out of consideration for the time being, we may attempt to determine *à priori* all possible problems whose different solutions possess this remarkable relation to each other. And now it appears that the choice of the groups of rotations of the regular polyedra as the object of the present investigation was no arbitrary one, but that these rotations and the corresponding groups of linear transformations of a single variable constitute a closed system by themselves, that, namely, every problem whose different solutions all proceed from any one of them by linear transformations is identical with one of the

problems of this system. In other words, every algebraic equation whose roots are all linear functions of any one of them is either an ikosaedron, an oktaedron, a tetraedron or a cyclical equation, or can be reduced to one of these by replacing the invariant Z by a linear function of itself, $Z' = \frac{aZ+b}{cZ+d}$, and z by $z' = \frac{\alpha z+\beta}{\gamma z+\delta}$. Or, again, every finite group of linear transformations of a single variable is holoedrically isomorphic, *i. e.* formally identical, with some one of the groups of rotations of the regular polyedra. This is the centre-point of this entire theory. *The problem with the various phases of which the present work deals is in no way an arbitrary one, but constitutes, in all its developments, a complete whole, perfectly defined in every direction by conditions and limitations inherent in its own nature.* At the same time, these conditions and limitations serve exactly to characterize the position of the present theory with respect to other related or more comprehensive theories. Thus, the finite ternary, quaternary, etc., linear groups admit of an analogous geometrical treatment; or, instead of increasing the number of variables, we may consider binary groups containing an infinite number of operations. On the other hand, I have already stated that the theory of the general algebraic equations of the first five degrees is identical with that of the rotations of the various regular polyedra and of the finite binary linear groups. Similarly, we might seek for corresponding relations between the theory of the higher equations and that of the ternary, quaternary, etc., groups. In this theory, which was proposed by Klein,* considerable progress has already been made. Thus, for the general equation of the sixth degree, the quaternary group of the transformations of the Borchardt moduli $\xi, \eta, \zeta, \vartheta$, and, for certain equations of the seventh and eleventh degrees, the ternary and quinary linear groups belonging to certain functions proposed by Klein,† play the same part as the groups of a single variable in the present case. The theory of the infinite groups leads to interesting developments, which will be considered later.

The problem with which the book before us has to deal is now completely defined on all sides. On the one hand, it is the examination of all groups of linear transformations of a single variable, which then, as we have just seen, is

* See in particular the article, "Ueber die Auflösung gewisser Gleichungen vom siebenten und achten Grade," Math. Ann. XV.

† *Ibidem.* See further the article in the same volume, "Ueber die Transformation elfter Ordnung," etc.

essentially identical with the determination of all groups of rotations of regular polyedra. This part of the theory, therefore, enjoys the joint benefit of the analytical and geometrical methods from the start. Having, through this examination obtained a complete understanding of the problem itself, we have, on the other hand, to determine in detail all other problems which are equivalent in their nature to the one considered. The development of the former of these two divisions of the subject leads at once to the consideration of the analytical nature of the actual solution of our problems in the third chapter, while the fourth chapter deals with the Galois theory of substitutions, to which the theory of mutually equivalent algebraic problems naturally belongs.

The third chapter is particularly instructive, on account not only of the insight which it gives into the nature of the actual solution of the problem and of the resulting extension of our knowledge of the theory of functions, but also because of the fundamental connection here exhibited between the theory of these new functions and the theory of the linear differential equations of the second and third orders. The introduction of these equations adds a new and most valuable instrument of research to those already at our disposal, and at the same time the dominant purpose of the book, the extension of the subject to meet all other related branches, is fully carried out in this direction.

The introduction of the differential equations into this theory is accomplished in a remarkably natural manner. We have a certain analytic or geometric configuration to which an invariant Z belongs, the elements of the configuration being denoted by η. One of these last being given, say η, all others are determined by $\xi_1 = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}$, where α, β, γ, δ are known quantities. If we differentiate this equation three times with respect to Z, and eliminate α, β, γ, δ from the resulting and original equations, we shall have a differential expression which, being independent of α, β, γ, δ, is independent of linear transformation. This expression is the equation $\frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2 = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2$, of which either side is such a "differential invariant." This Schwarzian derivative, as Cayley calls it; is therefore independent of the separate elements of the configuration, and depends only on the configuration as a whole; *i. e.* it is symmetrical in the roots of the corresponding equation, and is therefore a rational function of the invariant Z. The determination of the nature of this rational function is accomplished easily by aid of considerations from the theory of functions based on the

geometry itself. The differential equation then assumes the form

$$[\eta]_z = \frac{\nu_1^2-1}{2\nu_1^2(z-1)^2} + \frac{\nu_2^2-1}{2\nu_2^2 z^2} + \frac{\frac{1}{\nu_1^2}+\frac{1}{\nu_2^2}-\frac{1}{\nu_3^2}-1}{2(z-1)z},$$

where $[\eta]_z = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$, and ν_1, ν_2, ν_3 are characteristic numbers belonging to the special configurations already mentioned above, and denote in each case the number of points in the configuration which coincide in sets for the corresponding special values of Z. Thus, for the ikosaedron, $\nu_3 = 5$, corresponding to the five faces of the ikosaedron which meet at each vertex, and to the 5 points which consequently coincide when Z has the value belonging to the configuration of the vertices.

With this introduction of the differential equation of the third order, this portion of the theory is by no means complete. There remains to be noticed an important connection between the theory of this differential equation of the third order and those linear differential equations of the second order whose coefficients are rational functions of the independent variable. Thus if $y'' + py' + qy = 0$ be such an equation, and if we form the ratio of two particular solutions $\frac{y_1}{y_2}$, then, if the independent variable Z describe a closed path on its Riemann's surface, $\eta = \frac{y_1}{y_2}$ will become $\xi = \frac{\alpha y_1 + \beta y_2}{\gamma y_1 + \delta y_2}$. On account of this linear relation η satisfies a differential equation of the third order, whose left-hand side is identical with that of the equation already considered, while its right-hand side is a rational function of Z. It appears, furthermore, that not only can we always obtain from a linear differential equation of the second order another of the third order, but that we can also always accomplish the solution of the inverse problem. There is therefore a linear differential equation of the second order belonging to that of the third order which we have deduced from the theory of the linear groups. The importance of this equation of the second order depends on the fact that its solution is a special case of the Riemann function P, the connection of which with the hypergeometric series of Gauss is well known.

Returning to the differential equation of the third order, we have already seen how our problem admits of extension by the introduction of infinite groups of linear transformations of a single variable. In the fifth chapter will be found an account of the connection of this theory with that of the functions

of Schwarz. Thus, using the notation already explained, $\frac{1}{\nu_1}+\frac{1}{\nu_2}+\frac{1}{\nu_3}$ is always greater than 1 for all the problems which the present work treats. If now we give to ν_1, ν_2, ν_3 any integral values, we get a series of new functions. The geometrical representation of these on the sphere is very interesting, and is given in some detail, but of especial importance is the system of functions obtained by putting $\nu_1 = 2$, $\nu_2 = 3$, and giving ν_3 successively all integral values from 2 on. This gives in succession for $\nu_3 = 2, 3, 4, 5$ the diedron, the tetraedron, oktaedron and ikosaedron, and then an infinite series of transcendental functions ending with the elliptic modular functions for $\nu_3 = \infty$. Every one of these Schwarz functions, ν_1, ν_2, ν_3, is a rational function of every other, ν_1', ν_2', ν_3, if ν_1', ν_2', ν_3 are integral multiples of ν_1, ν_2, ν_3. For the present case, where $\nu_1 = 2$, $\nu_2 = 3$, every function of this series will therefore evidently be a rational function of $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = \infty$. It is exactly for this reason that the equation of the fifth degree, among others, can be solved by aid of the elliptic modular functions. By solution of such an equation, we mean simply the representation of its roots as rational functions of known quantities, in this case, of the known elliptic transcendents.

The actual solution of the problem having been thus discussed, and its bearing on other mathematical branches having been fully treated, it remains to determine the nature of all equivalent problems, *i. e.* of all problems whose solution is implicitly involved in that of the present one. For this purpose the Galois theory of substitutions is the efficient instrument, while it also secures us a completer knowledge of the essential nature of the internal structure of the problem. A development of this theory in its more essential and pertinent features occupies the fourth chapter of the book.* Starting from any algebraic equation of degree n, we may construct rational functions of its n roots $x_1 \ldots\ldots x_n$, and consider the effect which a permutation of these roots has on the form and value of these functions. A function may remain unchanged for all permutations of the roots, and is then symmetrical, or it may be changed by certain permutations and unchanged by others; and the number of permutations which leave it unchanged may vary from $n!$ in the case of the symmetrical functions to 0 for the utterly unsymmetrical ones such as $\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n$, where $\alpha_1 \ldots\ldots \alpha_n$ are all

* Here the reader will do well to take a course of parallel reading in the recent work of Netto mentioned in the footnotes of the Ikosaeder: "Substitutionentheorie und ihre Anwendungen auf die Algebra." Teubner, Leipzig.

different constants. All those permutations which leave any function unchanged constitute a group, since evidently, if the permutations σ_i and σ_j leave a function unchanged, then their successive application, denoted by $\sigma_i\sigma_j$, will also leave it unchanged. To every function then corresponds a group of substitutions, and, conversely, we can always find for every group a function which shall be unchanged by its operations. All functions belonging* to the same group constitute a family—Gattung†—which possesses the property that every function of the family is a rational function of every other one, with coefficients which are rational in the coefficient of the original equation. If we consider any function ϕ, and determine the corresponding group of substitutions, $1, \sigma_1 \ldots . \sigma_{r-1}$, then, if we apply any other substitution σ_i to ϕ, we shall get a new function, ϕ_2. Thus, $\sigma_i\phi_1 = \phi_2$. ϕ_2 is called conjugate to ϕ_1. Evidently there can be only a finite number of functions conjugate to a given one. If k be the number of values which a function takes when subjected to all possible permutations of the roots, and if r be the order of the group belonging to the function, *i. e.* the number of operations included in it, then $rk = n!$ Both r and k must therefore be factors of $n!$, which greatly restricts the number of possible groups and values of functions.

If $\phi_1 \ldots . \phi_k$ be all the values of a given function, then $\Sigma\phi_i$, $\Sigma\phi_i\phi_j$, etc., will all be symmetrical functions of the roots, and therefore rational functions of the coefficients of the original equation. We may therefore obtain an equation of degree k of which the coefficients are "rationally known" quantities, and of which $\phi \ldots . \phi_k$ are the roots. Such an equation is a *resolvent* of the original equation. If the group of ϕ_1 be $1, \sigma_i \ldots . \sigma_{r-1}$, and if σ_i converts ϕ_1 into ϕ_2, then the group of ϕ_2 will be $1 = \sigma_i 1, \sigma_i^{-1}, \sigma_i\sigma_1\sigma_i^{-4}, \ldots . \sigma_i\sigma_{r-1}\sigma_i^{-1}$. Here two important cases must be distinguished. Either the groups of $\phi_1 \ldots . \phi_k$ are distinct, or they coincide. If they are distinct, not only are the ϕ's rational functions of the x's by definition, but the x's are also rational functions of the ϕ's, since every permutation of the x's permutes the ϕ's also; so that $\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots . + \alpha_k\phi_k$ is a $n!$ valued function, and therefore every other rational function of the roots is a rational function of this. In this case, if we can solve the equation for the ϕ's, that of the x's is likewise solved, and *vice versa,* if the x equation is solved, the ϕ equation is solved with it; *i. e.* the two involve only the same irrationalities. But if the groups of the

* Any function unchanged by the operations of the group is said to *belong* to the group.

† See Kronecker's Festschrift, Crelle 92, for a further development of this idea. Cf. also in this connection Bachmann: Ueber Galois' Theorie der algebraischen Gleichungen, Math. Ann. XVIII.

$\phi_1 \ldots . \phi_k$ are coincident, in which case the group is said to be self-conjugate, the x's will not be rational functions of the ϕ's, and consequently the solution of the ϕ equation will not involve that of the x equation, but we still have to solve a second equation to determine the x's from the ϕ's. To compensate this, the ϕ equation is itself easier to solve, because of the relations which exist between its roots, every root being a rational function of every other root. In the former case, the given x equation and its resolvent present exactly the same problem; in the latter case, the introduction of the resolvent decomposes the problem into two simpler steps. The possibility of such a reduction depends, therefore, on the presence of a self-conjugate group. Such a group occurs for the oktaedron, and is characteristic of the equation of the fourth degree. For the fifth degree and the ikosaedron, on the contrary, no such group exists.

Through these considerations the question of the nature and of the determination of those problems which are equivalent to the given one may be regarded as settled. Equivalent algebraic problems are those which are resolvents of each other. To this idea of resolvent equations must be added that of the Galois group of an equation, when the theory will be complete in this direction. The notion of the group of an equation is essentially identical with the conception of what, in the problem of its solution, is to be looked upon as known or given, *i. e.* the *Rationalitätsbereich* of Kronecker. If the general equation of any degree be proposed for solution, evidently all that is known is the coefficients, and the solution of the equation consists in determining the roots in terms of these; but for special equations we may know other rational relations between the roots besides the symmetric ones. For instance, for the cyclical equation $x_n = 1$, where n is a prime number, we know that every root is a power of every other one except 1. Again, for the ikosaedron equation, we know that every root is a rational function of every other one. If, now, we construct every rational relation which exists between the roots in the form of an equation whose right-hand side is 0, the Galois group of the given equation is that group of permutations of the roots which leaves all these relations unchanged in value.*

The immediate application of this to the ikosaedron, oktaedron, etc., equations is obvious. In each case the group of the equation is composed of those

---

* I wish to append here a very concise definition of the Galois group of an equation given by Prof. Klein in one of his lectures: "The Galois group of an equation is the group of permutations of its roots which possesses the two properties—1st, that all corresponding rational functions of the roots are rationally known, and 2d, that it is the smallest group for which this is true."

permutations of the roots, or points of the configuration, which are produced by all the rotations of the corresponding polyedron. Thus the group of the ikosaedron equation is composed of 60 permutations of its roots, out of a possible 60!

The connection between this theory of the group of an equation and that of resolvents of the equation is complete when we notice that all resolvents of any equation have precisely the same group with the original equation, unless indeed the appearance of a self-conjugate sub-group should interfere with this. For, excluding this possibility, every permutation of the x's will produce a permutation of the ϕ's, so that, corresponding to the group of the x equation, there will exist a group of permutations of the ϕ's, which must then be the group of the ϕ equation. For every function which is rational in the x's is also rational in the ϕ's, and *vice versa*, and every function which, regarded as a function of the x's, is unchanged by a group of x permutations, will be, regarded as a function of the ϕ's, unchanged by the corresponding ϕ permutations.

All problems, therefore, equivalent to the given ones are resolvents of these, whereby they all possess a common characteristic—they all have the same group. To finish this portion of the theory, it only remains to construct all such resolvents. And here the actual connection of the ikosaedron and the other polyedra with the equations of the first five degrees, etc., is made obvious. *The general equations of degrees 2–5 and the cyclical equations are resolvents of the ikosaedron, etc., equations.* Thus the 60 rotations of the ikosaedron and the 60 even permutations of the roots of the general equations of the fifth degree are symbolically identical, *i. e.* holoedrically isomorphic; and similar relations hold for the simpler cases.

In the fourth chapter of the book, the more important resolvents are actually obtained. Among them, that of the sixth degree and the *Hauptgleichung* of the fifth degree are particularly to be noticed. The former is the equation for the transformation of the fifth degree of the elliptic functions. The latter will play an important part in the second portion of the book.

Within the brief space of a review, it is of course impossible even to touch upon many matters of great interest, for which reference must be made to the reviewed work itself. And having now developed the fundamental ideas of the first part of the "Ikosaeder" in considerable detail, I shall be obliged to forego giving any adequate account of the many important and elegant theories contained in the second part of the book. This, however, will by no means imply that this portion of the work is of less importance than the other; on the contrary, it

contains some of the most valuable features of the entire book. But, on the one hand, the structural nature of the problems dealt with here is already implicitly treated in the first part, and, on the other hand, the further developments which this last half of the book contains would scarcely be comprehensible without the actual study of the book itself. No account of the "Ikosaeder" would, however, be complete which did not call attention to the historical development of the theory of the equations of the fifth degree contained in the opening chapter of the second part, which may well be read before any other portion of the book. The abundant historical matter contained here, and the profusion of foot-notes throughout, are most valuable features, and constitute in themselves a complete encyclopædia of information. The methods of Tschirnhaus, Bring and Jerrard, and later those of Jacobi, Hermite, Brioschi and Kronecker, are all given in short sketches, with full references to the original works, which will materially assist the student in following the historical and philosophical development of these important systems. The reader will continually find himself referring to this, and to the fifth chapter of the first part, for the general direction and tendency of the broader phases of the theory.

Of the later chapters of the book, the fourth may be especially noted. It contains the treatment of the Jacobi modular equation of the sixth degree, which connects the work of Klein with the earlier theory of Kronecker, Brioschi, Hermite and others. I can only mention here that this theory is closely connected with that of a ternary group of linear transformations, just as the ikosaedron problem is related to that of the binary linear group of 60 transformations.

Before concluding, it remains to mention one matter which is of considerable general importance, and is characteristic of Klein's entire method. It is the exact meaning of the phrase "solution of an equation" in Klein's sense of the word. It will already be evident that this is something very different from the common conception of the words. Thus it is ordinarily said that the general equation of the fifth degree is solved by aid of the elliptic modular functions, whereas, from Klein's point of view, the introduction of these transcendental irrationalities is in no way essential to the theory, in fact, rather lies outside the region of the present work. According to Klein, an equation is to be regarded as solved when its complete *structural nature* is fully known. This includes the knowledge of the nature of all connection between the roots, all relations between different resolvent functions, all functional quantities which may be

regarded as known; in short, the entire Galois theory as applied to the case in hand; further, the nature of all those properties of the problem which are unchanged by groups of operations of any character, particularly those invariants which belong to groups of linear transformations; again, the complete knowledge of the nature of the actual solution as based on the method of the differential equations, which last then creates a new portion of the theory of functions; and finally, an adequate geometrical or hypergeometrical representation shall be found for all these characteristic properties. It is with the *implicit* nature of the equation that this theory deals, while the explicit form of the actual solution is a matter of comparative unimportance. The introduction of the elliptic transcendents in the solution of the equation of the fifth degree appears from this point of view, like the introduction of the trigonometric functions in the solution of the equation of the third degree, to belong rather to the theory of transcendental quantities than to the theory of equations.

Starting out from this broad conception, Klein has proposed a general theory of equations which shall contain in itself these various treatments. Thus, if we have to solve an equation, we first of all determine its Galois group. This having been done, the next step will be in each case to determine a finite group of linear transformations, of as few elements as possible, which shall be holoedrically isomorphic with the Galois group of the equation. The interpretation of these linear transformations as collineations of the corresponding space, and the determination of a corresponding invariant configuration in this space, form the basis of the geometrical treatment. Finally, a system of differential equations is to be obtained which are satisfied by the actual solution of the problem. The hyperelliptic functions may then be introduced as accessory irrationalities, just as the elliptic and trigonometric functions appear in the present theory.

CAMBRIDGE, September, 1886.